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TECHNICAL REPORT

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The following preprints in various stages of being prepared:

1. Reduction of 3x3 Polynomial Bundles and New Types of Integrable 3-Wave Interactions, by V.S. Gerdjikov and D.J. Kaup (to appear in the proceedings of the Como Conference of July 5-15, 1988).

2. Nonlinear Propagation of an Electromagnetic Pulse in a Two-Component Plasma, by D.J. Kaup and Ronald E. Kates (in final preparation).
   This manuscript describes how to correctly calculate the nonlinear coefficients in the case of an electromagnetic pulse propagating in a two-component plasma. We also demonstrate that other values given in the literature are incorrect. We correct the predictions for such electromagnetic propagation and discuss the astrophysical consequences.
   The above results have been presented as short talks at two meetings, the APS plasma physics meeting in Nov. 1988 and at the Grossman general relativity meeting in Australia in Aug. 1988.

3. The Nonlinear Propagation of a Relativistic Electromagnetic Pulse in Plasma, by Ronald E. Kates and D.J. Kaup (in preparation).
   This problem has become much more complex than first envisioned. In particular, the longitudinal electric field is found to be much larger than first estimated. In this limit, it seems that any charge separation leads to an intense longitudinal electrical field. The consequences of this is being explored numerically, in order to determine how to correctly formulate the expansion.

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Reduction Of 3 x 3 Polynomial Bundles

And New Types Of Integrable 3-Wave Interactions

by

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1. Our aim is to show that the group of reductions proposed by Mikhailov (A.V. Mikhailov, 1981) can be effectively used in constructing of new versions of integrable nonlinear evolution equations (NLEF) in 1 + 1 dimensions. We illustrate this by two examples, which lead to new integrable versions of the well known 3-wave interaction (V.E. Zakharov and S.V. Manakov, 1975) and (D.J. Kaup, 1976).

Let us consider a matrix Lax pair, first order in $\frac{d}{dx}$ and $\frac{d}{dt}$ of the form:

$$\left(i \frac{d}{dx} + U(x,t,\lambda)\right) \psi (x,t,\lambda) = 0 \qquad (1)$$

$$\left(i \frac{d}{dt} + V(x,t,\lambda)\right) \psi (x,t,\lambda) = 0 \qquad (2)$$

Following (A.V. Mikhailov, 1981) we will say, that it possesses a Z_N group of reductions if U and V satisfy the relations:

$$K^{-1}\, U(x,t,\lambda)\ K = U(x,t,\lambda\omega), \quad K^{-1}\, V(x,t,\lambda)\ K = V(x,t,\lambda\omega) \tag{3}$$

where K is a constant matrix such, that $K^N = I$ and $\omega = \exp\ (2\pi i/N)$. In what follows we shall also impose the involution (or Z_2 - reduction):

$$U^{\dagger}\ (x,t,\lambda) = B_1^{-1}\ U\ (x,\varepsilon\lambda)\ B_1, \tag{4}$$

and the same for $V(x,t,\lambda)$, where $B_1^{\dagger}\ B_1^{-1} = I$ and $\varepsilon = \pm 1$. We limit ourselves to the simplest possible case, when U and V are 3 x 3 matrix - valued functions, depending polynomially in λ.

2. As first example we shall consider U and V to be quadratic on λ:

$$U\ (x,t,\lambda) = \sum_{k=0}^{2}\ U_k\ (t,x)\lambda^k, \quad V(x,t,\lambda) = \sum_{k=0}^{2}\ V_k\ (x,t\ )\lambda^k \tag{5}$$

where

$$U_0\ (x,t) = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & u_{33} \end{bmatrix} \qquad U_1\ (x,t) = \begin{bmatrix} 0 & 0 & u_{13} \\ 0 & 0 & u_{23} \\ u_{31} & u_{32} & 0 \end{bmatrix} \tag{6}$$

$$V_0\ (x,t) = \begin{bmatrix} v_{11} & v_{12} & 0 \\ v_{21} & v_{22} & 0 \\ 0 & 0 & v_{33} \end{bmatrix} \quad V_1\ (x,t) = \begin{bmatrix} 0 & 0 & v_{13} \\ 0 & 0 & v_{23} \\ v_{31} & v_{32} & 0 \end{bmatrix} \tag{7}$$

and

$$-U_2 = \mathrm{diag}\,(a_1, a_2, a_3), \qquad -V_2 = \mathrm{diag}\,(b_1, b_2, b_3) \tag{8}$$

are constant diagonal matrices, whose entries are pairwise different: $a_1 \neq a_2 \neq a_3 \neq a_1$ and $b_1 \neq b_2 \neq b_3 \neq b_1$. Obviously, U and V defined by (5)-(8) satisfy (3) with N=2 and $K = \mathrm{diag}\,(1,1,-1)$. Imposing the involution (4) with $B_1 = \mathrm{diag}\,(1, \gamma, \alpha)$, α, γ - real constants, leads to:

$$\begin{aligned} u_{21} &= \gamma u_{12}^*, \quad u_{31} = \alpha u_{13}^*, \quad u_{32} = \frac{\alpha}{\gamma} u_{23}^* \\ v_{21} &= \gamma v_{12}^*, \quad v_{31} = \alpha v_{13}^*, \quad v_{32}^* = \frac{\alpha}{\gamma} v_{23}^* \\ u_{jj} &= u_{jj}^*, \quad v_{jj} = v_{jj}^*, \quad a_j^* = a_j, \quad b_j^* = b_j \end{aligned} \tag{9}$$

After some calculations we find, that the compatibility condition for (1),(2) gives us the following expressions for v_{ij} in terms of u_{ij}:

$$\begin{aligned} v_{13} &= \eta_{13}\, u_{13}, \qquad v_{23} = \eta_{23}\, u_{23} \\ v_{12} &= \eta_{12}\, u_{12} + \frac{\alpha\kappa}{\gamma} u_{13}\, u_{23}^* \end{aligned} \tag{10}$$

where

$$\eta_{2j} = \frac{b_i - b_j}{a_i - a_j}, \quad \kappa = \frac{\eta_{23} - \eta_{13}}{a_1 - a_2} \tag{11}$$

and the following NLEE for u_{ij} :

$$i\,(\partial_t - \eta_{13}\,\partial_x)\,u_{13} = (\eta_{23} - \eta_{12})\,u_{12}\,u_{23} - \frac{\alpha\kappa}{\gamma}\,u_{13}\,|u_{23}|^2$$

$$+ u_{13}\,[\eta_{13}\,(u_{11} - u_{22}) + v_{33} - v_{11}]$$

$$i(\partial_t - \eta_{23}\,\partial_x)\,u_{23} = \gamma\,(\eta_{13} - \eta_{12})\,u_{13}\,u_{12}^* - \kappa\alpha\,|u_{13}|^2\,u_{23} \tag{12}$$

$$+ u_{23}\,[\eta_{23}\,(u_{22} - u_{33}) + v_{33} - v_{22}]$$

$$i(\partial_t - \eta_{12}\,\partial_x)\,u_{12} = \frac{\alpha\kappa}{\gamma}\,[i\partial_x(u_{13}\,u_{23}^*) + u_{13}\,u_{23}^*\,(u_{11} - u_{22})]$$

$$+ u_{12}\,[\eta_{12}\,(u_{11} - u_{22}) + v_{22} - v_{11}]$$

In (10) ((12)) we have written down only the relations (NLEE's) for v_{ij} (u_{ij}) with $i < j$; the corresponding ones with $i > j$ are obtained from them by complex conjugation and the use of (9). As regards the diagonal elements u_{ij} and v_{ij}, they satisfy:

$$i(v_{jj,x} - u_{jj,t}) + F_{jj}(x,t) = 0$$

$$F_{11} = -F_{22} = \alpha\,\kappa\,(u_{12}\,u_{13}^*\,u_{23} - u_{12}^*\,u_{13}\,u_{23}^*);\; F_{33} = 0 \tag{13}$$

We can fix up the diagonal terms u_{jj} and v_{jj} by choosing the gauge of the Lax pair. There are many possibilities to do this:

$$u_{jj} = \mu_j |u_{13}|^2 + \gamma\nu_j |u_{23}|^2$$

$$v_{jj} = \mu_j n_{13} |u_{13}|^2 + \gamma\nu_j n_{23} |u_{23}|^2 \qquad j = 1,2,3 \tag{14}$$

Then equations (13) are direct consequence of (12), if the constants μ_j, ν_j are related by:

$$\mu_j = \frac{a_2 - a_3}{a_1 - a_3}\nu_j + \theta_j , \quad \theta_1 = -\theta_2 = -\frac{\alpha}{\gamma(a_1 - a_3)} , \ \theta_3 = 0 \tag{15}$$

Note that only differences of μ_j and ν_j occurs in (12), and by a phase transformation on u_{ij} and v_{ij}, one can also transform some of the constants in (15) to zero. For example, one may easily phase transform all three μ_j's to be equal or all three ν_j's to be equal. But because of (13), with F_{jj} nonzero in general, one may never phase transform all three μ_j's or all three ν_j's to be zero.

Thus the first two equations in (12) contain in addition to the usual bilinear in u_{ij} nonlinearities, also cubic terms. In the third equation in (12) the usual bilinear in u_{ij} term appears under an x-derivative; here we also have cubic terms.

Note, that both reductions on U and V commute between themselves, so the total reduction group is $Z_2 \otimes Z_2$.

As an example, let us choose $a_3 = 0$, $a_2 = -a_1$, $b_1 = b_2 = 0$, $b_3 = -ca_1$, $a_1 = c/\kappa$, $u_{13} = E$, $u_{23} = F$, $u_{12} = N$, $\gamma = -1$, $\nu_2 = \nu_1 + 1/c$, $\nu_3 = \nu_2 - 3/(2c)$, and $\alpha\kappa = 1$, then (12) reduces to

$$i\,(\partial_t - c\,\partial_x)\,E = -\,cNF$$

$$i\,(\partial_t + c\partial_x)\,F = -\,cN^*E \tag{16}$$

$$i\partial_t N = -\,i\,\partial_x(F^*E) + F^*E\,(|E|^2 - |F|^2)\,\frac{1}{c} + N\,(|E|^2 + |F|^2)$$

These equations remind one of two counterpropagating electromagnetic beams, E and F, interacting via some density modulation, N. And these are another example of integrable interactions between long waves and short waves (D.J. Benny, 1977) and (Alan C. Newell, 1978).

3. Now consider U and V to be cubic in λ:

$$U\,(x,t,\lambda) = \sum_{k=0}^{3} U_k\,(x,t)\lambda^k\ ,\quad V\,(x,t,\lambda) = \sum_{k=0}^{3} V_k\,(x,t)\,\lambda^k \tag{17}$$

and such, that, U_0, U_3, V_0 and V_3 are diagonal:

$$U_0 = \mathrm{diag}\,(u_{11}, u_{22}, u_{33}),\quad -U_3 = +\,\mathrm{diag}\,(a_1, a_2, a_3)$$

$$V_0 = \mathrm{diag}\,(v_{11}, v_{22}, v_{33}),\quad -V_3 = \mathrm{diag}\,(b_1, b_2, b_3) \tag{18}$$

where a_j, b_j are the same constants as in (8) above. The matrices U_1, U_2, V_1, V_2 are given by:

$$U_1(x,t) = \begin{bmatrix} 0 & u_{12} & 0 \\ 0 & 0 & u_{23} \\ u_{31} & 0 & 0 \end{bmatrix}, \quad U_2(x,t) = \begin{bmatrix} 0 & 0 & u_{13} \\ u_{21} & 0 & 0 \\ 0 & u_{32} & 0 \end{bmatrix} \tag{19}$$

$$V_1(x,t) = \begin{bmatrix} 0 & v_{12} & 0 \\ 0 & 0 & v_{23} \\ v_{31} & 0 & 0 \end{bmatrix}, \quad V_2(x,t) = \begin{bmatrix} 0 & 0 & v_{13} \\ v_{21} & 0 & 0 \\ 0 & v_{32} & 0 \end{bmatrix} \tag{20}$$

With this choice U and V automatically satisfy the reduction condition (3) with $N=3$ and $K = \mathrm{diag}\,(1, \omega, \omega^2)$, $\omega = \exp(2\pi i/3)$. We can also impose the involution (4)

with $\varepsilon = -1$ and $B_1 = \begin{bmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 1 & 0 \\ \varepsilon_1 & 0 & 0 \end{bmatrix}$, $\varepsilon_1 = \pm 1$, which gives:

$$\begin{aligned} &u_{12} = -\varepsilon_1 u_{23}^*, \qquad && u_{31}^* = -u_{31}, \qquad && u_{13}^* = u_{13} \\ &u_{21} = \varepsilon_1 u_{32}^*, && u_{11}^* = -u_{33}, && u_{22}^* = -u_{22} \\ &a_3^* = -a_1, && a_2^* = -a_2 && \end{aligned} \tag{21}$$

and analogous relations for v_{ij} and b_j. The compatibility condition now gives:

$$v_{12} = -\eta_{12} u_{12} + \kappa\epsilon_1 u_{13} u_{21}^* ; \qquad v_{13} = \eta_{13} u_{13}$$

$$v_{31} = \eta_{13} u_{31} + \kappa\epsilon_1 |u_{21}|^2 ; \qquad v_{21} = \eta_{12} u_{21} \tag{22}$$

where $_{ij}$ and κ are expressed through a_j, b_j as in (11). The corresponding NLEE have the form:

$$i(\partial_t - \eta_{13}\partial_x)u_{13} = \epsilon_1 (\eta_{12} - \eta_{23})|u_{12}|^2 + \kappa u_{13} (u_{12} u_{21} + u_{12}^* u_{21}^*)$$

$$+ u_{13} [\eta_{13} (u_{11} - u_{33}) + v_{33} - v_{11}]$$

$$i(\partial_t - \eta_{12}\partial_x)u_{21} = -\epsilon_1 (\eta_{13} - \eta_{23}) u_{12}^* u_{31} - \kappa u_{21} (u_{12}^* u_{21}^* + u_{13} u_{31})$$

$$+ u_{21} [\eta_{12} (u_{22} - u_{11}) + v_{11} - v_{22}]$$

$$i(\partial_t - \eta_{12}\eta_x) u_{12} = i\kappa\epsilon_1 (u_{13} u_{21}^*)_x + \kappa\epsilon_2 u_{13} u_{21}^* (u_{11} - u_{22})$$

$$+ u_{12} [\eta_{12} (u_{11} - u_{22}) + v_{22} - v_{11}]$$

$$i(\partial_t - \eta_{13}\partial_x) u_{31} = i\kappa\epsilon_1 (|u_{21}|^2)_x + \kappa\epsilon_1|u_{21}|^2 (u_{33} - u_{11})$$

$$+ u_{31} [\eta_{13} (u_{33} - u_{11}) + v_{11} - v_{33}] \tag{23}$$

For the diagonal elements u_{jj}, v_{jj} we get

$$u_{jj,t} - v_{jj,x} = 0 , \quad j = 1,2,3 \tag{24}$$

If we fix the gauge in the simplest possible way, choosing $v_{jj} = u_{jj} = 0$, then what we get is a modification of the 3-wave equations with additional cubic nonlinearities in the first two equations in (23); the nonlinearities in the two last equations of (23) just acquire additional x-derivative.

However, we can choose another gauge by requiring that the corresponding linear problems (1), (2) become equivalent to a Riemann-Hilbert problem with canonical normalization. This leads to:

$$u_{11} = \frac{u_{12}\, u_{21}}{a_1 - a_2} + \frac{u_{13}\, u_{31}}{a_1 - a_3} - \frac{\varepsilon_1\, u_{13}\, |u_{21}|^2}{(a_1 - a_3)(a_1 - a_2)}$$

$$u_{22} = -\frac{u_{12}^{*}\, u_{21}^{*}}{a_2 - a_3} - \frac{u_{12}\, u_{21}}{a_1 - a_2} + \frac{\varepsilon_1\, u_{13}\, |u_{21}|^2}{(a_1 - a_s)(a_2 - a_3)} \qquad (25)$$

and

$$v_{11} = \frac{\eta_{12}\, u_{12}\, u_{21}}{a_1 - a_2} + \frac{\eta_{13}\, u_{13}\, u_{31}}{a_1 - a_3} + \theta_{11}\, \varepsilon_1\, u_{13} |u_{21}|^2$$

$$v_{22} = -\frac{\eta_{23}\, u_{12}^{*}\, u_{21}^{*}}{a_2 - a_3} - \frac{\eta_{12}\, u_{12}\, u_{21}}{a_1 - a_2} + \theta_{22}\, \varepsilon_1\, u_{13} |u_{21}|^2$$

$$\theta_{11} = \frac{3a_1 \kappa - \eta_{23}}{(a_1 - a_2)(a_1 - a_3)}, \qquad \theta_{22} = \frac{-3a_2 \kappa + \eta_{13}}{(a_1 - a_2)(a_2 - a_3)} \qquad (26)$$

In deriving the last line of (26) we have used also, that $\sum_{j=1}^{3} a_j = 0$; u_{33} and v_{33} defined by $\sum_{j=1}^{3} u_{jj} = \sum_{j=1}^{3} v_{jj} = 0$. From (27), (26) we see, that such choice of the gauge leads to an additional quartic and quintic nonlinearities in the NLEE (23).

At the end we give the explicit form of the NLEE (23) in the simplest possible case, when $u_{jj} = v_{jj} = 0$ and moreover $\eta_{12} = \eta_{23} = c =$ real and $\eta_{13} = 0$. Denoting

$$
\begin{aligned}
u_{12} &= E(x,t)\,, \quad u_{13} = n(x,t)\,, \quad u_{23} = -\,\varepsilon_1\, E^*(x,t) \\
u_{21} &= F(x,t)\,, \quad u_{31} = iN(x,t)\,, \quad u_{32} = \varepsilon_1\, F^*(x,t)
\end{aligned}
\qquad (27)
$$

we get the following system for the two real $n(x,t)$, $N(x,t)$ and the two complex-valued functions $E(x,t)$ and $F(x,t)$:

$$
\begin{aligned}
\partial_t n &= g\, n\, (E^*F^* + EF) \\
\partial_t N &= g\, \varepsilon_1\, \partial_x\, (F^*F) \\
\partial_t F &= -g\, E^*\, (F^*F) - i\, g\, N\, n\, F \\
\partial_t E &= i\, \varepsilon_1\, g\, \partial_x\, (n\, F^*)
\end{aligned}
\qquad (28)
$$

Here $g = -i\kappa$ is a real-valued constant.

The question for possible physical applications of these equations is open.

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References

D.J. Kaup (1976). The three wave interaction - a non-dispersive phenomenon. Studies in Applied Mathematics 55, 9-44.

A.V. Mikhailov (1981). The reduction problem and the inverse scattering method. Physica D, 3D, 73-117.

V.E. Zakharov and S.V. Manakov (1975). On the theory of resonant interaction of wave packets in nonlinear media. Journal of Experimental and Theoretical Physics, (USSR) 69, 1654-1673.

D.J. Benny (1977). A General Theory for Interactions Between Short and Long Waves. Studies in Applied Mathematics 56, 81-94.

Alan C. Newell (1978). Long Waves-Short Waves; a Solvable Model. SIAM Journal of Applied Mathematics 35, 650-664.